◐ 14 ◑

Curry's Lively Bird Forest

When Inspector Craig reached Curry's Forest, the first thing he did was to interview the resident bird sociologist, whose name, curiously enough, was Professor Byrd.

"In this forest," said Byrd, "certain birds sing on certain days. It has been my purpose to determine which birds sing on which days. So far, I have not been able to come to a definite solution. I have been looking for one unifying principle—one general law that would enable me to decide which birds sing on which days. Over a period of many years I have gathered an enormous amount of statistical data; I have amassed tens of thousands of facts, and aided by a high-speed computer, I have been able to amalgamate all these facts into four general laws. These four laws give me *partial* information, but I cannot see how I can determine from them exactly which birds sing on which days. I have the feeling that there should be *just one general law* that would unify these four laws—much as Newton's universal law of gravitation unified Kepler's three laws of planetary motion. But I have not been able to find it. I wonder if you could help me."

"I'll do what I can," said Craig. "What are the four laws?"

"Well, we have here a very special bird P. I do not know its species, nor does it matter. The important thing is that for any bird x and any bird y, whether the same as x or different, the following laws hold:

Law 1: If y sings on a given day, then Pxy sings on that day.

Law 2: If x doesn't sing on a given day, then Pxy sings on that day.

Law 3: If the bird x and the bird Pxy *both* sing on a given day, then y sings on that day.

Law 4: For every bird x there is a bird y such that y sings on those and only those days on which Pyx sings.

"Those are my four laws," said Byrd. "Can you unify them into one grand law?"

"I'll have to think about it," said Craig, rising. "I'll be back tomorrow and tell you if I've found anything significant."

Craig went back to the inn in which he was staying and devoted some time to the matter. At one point he burst out laughing. "What a ridiculously simple law!" thought Craig. "How could Byrd have overlooked it all these years? I think tomorrow I'll have a bit of fun with him."

Craig visited Byrd the next day.

"I've solved your problem," said Craig. "From your four laws I have been able to deduce one very general law, which in turn easily explains why the four particular laws are true."

"Wonderful!" cried Byrd. "What is this general law?"

"Rather than tell you outright, I'll give you a hint. It follows from your laws that all sparrows here sing on Tuesdays."

"Amazing!" cried Byrd. "It so happens that all sparrows here do sing on Tuesdays, but how could you have deduced this from what I have told you? I haven't said anything about sparrows or Tuesdays; what's so special about sparrows and Tuesdays?"

"Nothing special about either," replied Craig, "and this very fact should give you a hint as to what my general law is."

Byrd sank back in puzzled thought.

"Don't tell me," he said at last, "that *all* birds here sing on *all* days!"

"Exactly!" said Craig.

"Fantastic!" cried Byrd. "Why didn't this possibility ever occur to me before? But I still don't completely understand. Why does it follow from the four laws I have given you that all birds here sing on all days?"

• 1 •

Why does it follow?

• 2 •

Suppose we are given Byrd's first three laws, but instead of Byrd's fourth law, we are given that the forest contains a lark. Does it then follow that all the birds sing on all days? Suppose that instead of being given a lark, we are given that there is a cardinal; would it then follow that all the birds sing on all days? Suppose we are given *both* a lark *and* a cardinal; does it then follow that all the birds sing on all days?

• 3 •

Again suppose we are given Byrd's first three laws, but we are not given the fourth. Can you find a *single* combinatorial bird whose presence would imply that all the birds sing on all days?

Discussion (to be read *after* the reader has gone through the solutions of the last three problems): The above problems are all closely related to a famous result known as *Curry's paradox*. Suppose that instead of talking about birds, we talk about propositions. And suppose that instead of talking about a bird singing or not singing on a given day, we talk about a proposition being true or false; every proposition is one or the other, but not both. For any proposition x and y, let Pxy be the proposition that either x is false or y is true, or what is the same thing, if x is true, then so is y. Then Byrd's first three laws correspond to the following three elementary laws of logic:

Law 1: If y is true, then Pxy is true.

Law 2: If x is false, then Pxy is true.

Law 3: If x and Pxy are both true, so is y.

Law 1 says that if y is true, then either x is false or y is true, which is obvious, because if y is true, then regardless of whether x is true or false, at least one of the propositions x and y is true—namely y. Law 2 says that if x is false, then either x is false or y is true; this is again obvious. As to Law 3, suppose x and Pxy are both true. Since Pxy is true, then either x is false or y is true. The first alternative—x is false—doesn't hold, since x is true, so the second alternative must hold—y is true.

Now, suppose we add the following law, which corresponds to Byrd's fourth law:

Law 4: For any proposition x there is a proposition y such that the proposition y and the proposition Pyx are either both true or both false. That is, the bird y and the bird Pyx either both sing or both do not sing on a given day.

What happens if we add Law 4 to the other three laws of logic? We then get a paradox, because from the four laws 1, 2, 3, and 4 we can prove that *all* propositions are true, in exactly the same way as we proved from Byrd's four laws that all the birds sing. Obviously it is *not* the case that all propositions are true, and so the addition of Law 4 to the other three laws creates an absurdity. This is Curry's paradox.

It should be pointed out that Byrd's four laws *as applied to birds*, which Byrd did, doesn't create any paradox; it merely leads to the conclusion that all birds of the forest sing on all days, and there is no reason why this can't be. It is only when the four laws are applied—or, I should say, "misapplied"—to *propositions* in the way indicated above that a genuine paradox arises.

Suppose we now consider an arbitrary collection of entities called *objects*, and suppose we have a certain operation which applied to object x and object y yields a certain object xy. We then have what is called an *applicative system*, in which the object xy is called the result of *applying* x to y. We have been studying applicative systems for the last several chapters; our "objects" were birds and we took xy to be the response of x to y. Combinatory logic studies applicative systems with certain special properties, among which is the existence of various combinators, including C, which we have called a *cardinal*, and L, which we have called a *lark*. Now, suppose the "objects" we are studying include all propositions, both true and false, as well as other objects, the *combinators*. Suppose we have an object P such that for any *proposition* x and y, the object Pxy is the proposition that either x is false or y is true. If x and y are not both propositions, then Pxy is still a well-defined object and may or may not be a proposition, depending on the nature of x and y. Laws 1, 2, and 3, of course, hold, *provided x and y are propositions*! Also, assuming C and L are present, given any object x, there must be an object y such that y = Pyx, as we saw in the solution to Problem 2. In particular, given any *proposition* x there must be an *object* y such that y = Pyx, but this y needn't be a proposition! In fact, y *can't* be a proposition, because if it were, Pyx would also be a proposition and the same proposition as y, which would mean that Law 4 would hold and we would again run into Curry's paradox. So the way out of the paradox is to realize that given a proposition x, although the axioms of combinatory logic imply that there is some *object* y such that y = Pyx, such a y cannot be a proposition. Some of the earlier systems, which attempted to combine the logic of propositions with combinatorial logic, were careless on this point and so the systems turned out to be inconsistent. But, as Haskell Curry pointed out, the paradoxes were not the fault of combinatory logic itself, they were the result of the misapplication of combinatory logic to the logic of propositions.

SOLUTIONS

1 · Let us first observe that it follows from Byrd's first two laws that if y sings on all days on which x sings, then the bird Pxy must sing on all days. *Reason*: Suppose that y sings on all days on which x sings. Now consider any day. Either x sings on that day or it doesn't. If x doesn't, then Pxy sings on that day by Byrd's second law. Now suppose x does sing on that day. Then y also sings on that day (because of the assumption that y sings on all days on which x sings), and hence Pxy must sing on that day by Byrd's first law. This proves that regardless of whether x does or doesn't sing on that day, the bird Pxy sings on that day. Hence Pxy sings on all days.

Now we will show that given any bird x, it sings on all days. Well, by Law 4, there is a bird y that sings on those and only those days on which Pyx sings. Now, consider any day on which y sings. Pyx also sings on that day, by Law 4, and since y sings on that day, then x sings on that day, by Law 3. This proves that x sings on all days on which y sings, and hence Pyx sings on all days, by the argument of the preceding paragraph. Then, since y sings on the same days as Pyx, the bird y sings on all days. Therefore, on any day at all, the bird y and the bird Pyx both sing, hence x also sings on that day, by Law 3. This proves that x sings on all days.

2 · If we are given just L alone or just C alone, then I see no way of proving that all the birds sing on all days, but if we are given *both* C and L, then we can derive Law 4 as follows:

Since the lark L is present, then every bird is fond of at least one bird; we recall that x is fond of Lx(Lx). Now take any bird x. Then the bird CPx is fond of some bird y, which means that CPxy = y, hence y = CPxy. But also CPxy =

Pyx, and so y = Pyx. Then of course y sings on the very same days as Pyx, because y *is* the bird Pyx! Thus Law 4 follows.

3 • Suppose that instead of being given the presence of both C and L, we are given that there is a bird A present satisfying the condition Axyz = x(zz)y. Then for any birds x and y, APxy = P(yy)x. Hence APx(APx) = P(APx(APx))x, and so y = Pyx, where y is the bird APx(APx).

SOME BONUS EXERCISES

Exercise 1: Suppose we are given Byrd's first three laws but not the fourth. Prove that for any bird x, y, and z the following facts hold:

a. Pxx sings on all days.
b. If Py(Pyx) sings on all days, so does Pyx.
c. If Pxy and Pyz sing on all days, so does Pxz.
d. If Px(Pyz) sings on all days, so does P(Pxy)(Pyz).
e. If Px(Pyz) sings on all days, so does Py(Pxz).

Exercise 2: Suppose we have a bird forest in which certain birds are called *lively*. We are not given a definition of *lively*, but we are told that there is a bird P such that the following three conditions hold:

a. For any birds x and y, if Px(Pxy) is lively, so is Pxy.
b. For any birds x and y, if x and Pxy are both lively, so is y.
c. For any bird x there is a bird y such that the birds Py(Pyx) and P(Pyx)y are both lively.

Show that all the birds of the forest are lively.

Exercise 3: The above exercise contains a somewhat stronger result than that of Problem 1 concerning Curry's Forest. Define a bird of Curry's Forest to be *lively* if it sings on all days. Then show that Byrd's four laws imply that the three conditions above all hold. It then follows from Exercise 2 that all the birds of the forest sing on all days, hence the solution of Problem 1 is a corollary of Exercise 2.

◐ 15 ◑
Russell's Forest

The next forest visited by Inspector Craig was known as Russell's Forest. Almost as soon as Craig arrived, he had an interview with a bird sociologist named McSnurd. He told McSnurd about his experiences in the last forest.

"As far as I know," said McSnurd, "we have no bird here satisfying Byrd's four laws. What we do have is a special bird a such that for any bird x, the bird ax sings on those and only those days on which xx sings. Also, for any bird x, there is a bird x′ such that for every bird y, the bird x′y sings on those and only those days on which xy does not sing. I hope this information will prove helpful."

Inspector Craig listened to this report with interest. Later that evening, sitting quietly in his room at the Bird Forest Inn, Craig reviewed the report and realized that McSnurd wasn't a very good observer, because the two facts he reported were logically incompatible.

• 1 •

Why is McSnurd's report inconsistent?

Solution: It is best that we give the solution immediately. Suppose McSnurd's report were true. We consider the bird a satisfying the condition that for every bird x, ax sings on just those days on which xx sings. Then according to McSnurd's second statement, there is a bird a′ such that for every bird x, a′x sings on just those days when ax doesn't sing. But the days when ax doesn't sing are just those days on which xx doesn't sing (because ax sings on the very same days as xx), and so we have a bird a′ such that for every bird x, a′x sings on just those days when xx doesn't sing. Since this holds for *every* bird x, it holds when x is the bird a′, and so a′a′ sings on those and only those days on which a′a′ doesn't sing, which is obviously a contradiction.

This paradox is a genuine one and is like the paradox of the barber who shaves those and only those people who don't shave themselves, or like Russell's famous paradox of the set that contains as members those and only those sets that do not contain themselves as members. Such a set would contain itself as a member if and only if it doesn't.

2 • A Follow-up

Inspector Craig was distinctly dissatisfied with Professor McSnurd, and so he asked one of the more learned inhabitants whether there were any other bird sociologists residing in Russell's Forest.

"This I do not know," was the reply, "but I do know that there is a *meta-bird-sociologist* in this forest; his name is Professor MacSnuff."

"Just what is a meta-bird-sociologist?" asked Craig in amazement.

"A meta-bird-sociologist is one who studies the sociology of bird sociologists. Professor MacSnuff is the leading authority, not on bird sociology, about which he knows nothing, but on bird sociologists. He is familiar with all the bird sociologists in the world, hence he should know which ones reside here. I suggest you contact him."

Craig expressed his thanks and then arranged an interview with MacSnuff.

"Yes, there is another bird sociologist here," said

MacSnuff. "His name is also McSnurd. He is a brother of the McSnurd you have already interviewed."

Craig was delighted, and arranged an interview with this other McSnurd.

"Ah, yes," said McSnurd. "My brother is not always accurate; he should not have told you what he did. What he *should* have said is that there is a bird N here such that for any bird x, the bird Nx sings on those and only those days on which x does not sing. Also, this forest contains a sage bird, if that will help."

Inspector Craig thanked him and left. "Oh drat!" said Craig to himself a moment later. "This McSnurd is as bad as his brother!"

How did Craig know this?

3 • A Second Follow-up

"Isn't there any *competent* bird sociologist in this forest?" Craig asked MacSnuff on his second visit.

"There is only one more bird sociologist here," said MacSnuff. "His name is also McSnurd and he is the brother of the other two McSnurds."

None too hopefully, Craig arranged an appointment with the remaining McSnurd.

"Ah, yes," said the third McSnurd. "Neither of my brothers is very good at either observing or reasoning. The last McSnurd you saw was right about the sage bird; I have seen one here myself. But he was wrong about the bird N; what he *should* have told you is that there is a bird A here such that for any birds x and y, the bird Axy sings on those and only those days on which neither x nor y sings. Now you shouldn't get into any trouble."

Does the third McSnurd's story hold water?

SOLUTIONS

2 · Suppose McSnurd's report were correct. Then for every bird x, $Nx \neq x$—that is, Nx is unequal to x—because Nx sings on just those days on which x doesn't. But since a sage bird is present, then every bird is fond of some bird, hence N is fond of some bird x, which means that $Nx = x$. This is a contradiction.

3 · The contradiction involved in this report is a bit more subtle and more interesting! Let us suppose the report is true. Take any bird x. Since there is a sage bird, then Ax, like every other bird, is fond of some bird y, so $Axy = y$. Thus y sings on those and only those days on which neither x nor y sings. If y ever sang on a given day, then neither x nor y would sing on that day, which means that y wouldn't sing on that day and we would have a contradiction. Therefore y never sings at all. Now, suppose there were some day on which x doesn't sing. Then neither x nor y sings on that day, hence Axy *does* sing on that day, and y sings on that day, contrary to the already proved fact that y never sings. Therefore x must sing on all days. And so we have proved that *every* bird x sings on all days, yet we have shown that for every bird x there is some bird y that never sings. This is obviously a contradiction.

◐ 16 ◑

The Forest Without a Name

Unable to find any reliable bird sociologist in Russell's Forest, Craig left it in disgust. Over the next several days he wended his weary way to the forest of this story.

For the first few days of his sojourn here, he was unaccountably sad. He could not analyze just *why* he was sad, but the fact remained that he was sad. "Could it be the disappointing results of my visit to the last forest?" thought Craig. "No," he concluded, "something else is also wrong, but I can't put my finger on just what the something is!"

Craig brightened somewhat when he heard that the bird sociologist of this forest was the eminent Professor McSnurtle. Though a cousin of the McSnurd brothers, McSnurtle was known to be thoroughly reliable. Craig had read about him back home in the *Encyclopedia of Bird Sociology*, and the one thing that was emphasized was that McSnurtle *never made mistakes*! Craig was granted an interview.

"We have a special bird e," said McSnurtle. "After years of research, I have established the following four laws concerning e.

Law 1: For any birds x and y, if exy sings on a given day, so does y.

Law 2: For any birds x and y, the bird x and the bird exy never sing on the same day.

Law 3: For any birds x and y, the bird exy sings on all days on which x doesn't sing and y does sing.

Law 4: For any bird x there is a bird y such that y sings on the same days as eyx.

"That," said McSnurtle proudly, "neatly sums up all I know about the singing habits of the birds of this forest."

Inspector Craig pondered this analysis well. At one point he could not completely suppress a slightly disdainful expression.

"What's wrong?" asked McSnurtle, who was quite a sensitive individual. "Have you found an inconsistency in my statements?"

"Oh, no," replied Craig. "I thoroughly trust your reputation for complete accuracy. Only there is one question I would like to ask you: Have you ever heard any birds in this forest sing at all?"

Professor McSnurtle wracked his brain for several minutes. "Come to think of it, I don't believe I ever have!" he finally replied.

"And I'm afraid you never will," said Craig, rising. "You could have stated your laws more succinctly still by combining them into the one simple law: *None of the birds of this forest ever sing*. I see now why I've felt so sad here!"

How did Craig realize this?

SOLUTION

This is essentially Problem 1 of Curry's Forest again. Let us say that a bird is *silent* on a given day if it doesn't sing on that day. Then McSnurtle's four laws can be equivalently stated as follows:

Law 1: If y is silent on a given day, then exy is silent on that day.

Law 2: If x is not silent on a given day, then exy is silent on that day.

Law 3: If the bird x and the bird exy are both silent on a given day, then y is silent on that day.

Law 4: For any bird x there is a bird y such that y is silent on those and only those days on which eyx is silent.

And so Byrd's four laws for P hold for e if we simply replace "sings" by "is silent." Then the same argument showing that all the birds of Curry's Forest sing on all days shows that all the birds of this forest are silent on all days.

Epilogue: Many years later, the Forest Without a Name (which actually did have a name of a paradoxical sort) came to be known as the Forest of Silence.

◐ 17 ◑
Gödel's Forest

Craig's next adventure was far more delightful and also highly informative. After leaving the Forest Without a Name, he found himself in the lovely forest of this chapter. The first thing he noticed was the abundance of birds in song. They sang so beautifully—just like nightingales! The bird sociologist of this forest was a certain Professor Giuseppe Baritoni, who himself had been an excellent singer in his day.

"Now in *this* forest," explained Baritoni, "we do not regard it of much importance which birds sing on which days; the important question is which birds can sing at all! Not all birds of this forest can sing. We have plenty of nightingales, and they all sing, as you may have gathered."

"Oh, yes," said Craig. "As a matter of fact, all the birds I have heard so far have sounded to me like nightingales. Are nightingales the *only* birds here who sing, or are there others?"

"Ah, a most interesting question!" replied Baritoni. "Unfortunately we have not found the answer. The only birds I have heard sing here are nightingales, and I don't know anyone who has heard a singing bird that is not a nightingale. Still, that's not conclusive evidence that nightingales are the only singing birds of this forest; it may be that there is some bird not yet discovered that sings but is not a nightingale. It would be most interesting if there were!

"As a matter of fact, a logician from the Institute for Advanced Study in Princeton once visited this forest many years ago, and when I told him some of the singing laws of this forest, he conjectured that it *should* be decidable on the basis of these laws whether or not there was such a bird. Unfortunately, he left one day quite suddenly, and I forgot his name. I have never heard from him since."

"What *are* these laws?" asked Craig with enormous interest.

"Well," explained Baritoni, "the first interesting thing about this forest is that all the birds are married. For any bird x, by x′ I mean the mate of x. The interesting thing is that for any birds x and y, the bird x′y sings if and only if xy does not sing.

"The second interesting thing is that every bird x has a distinguished relative x* called the *associate* of x. The bird x* is such that for every bird y, the bird x*y sings if and only if x(yy) sings.

"The third thing is that there is a special bird $\mathcal{N}$ such that whenever you call the name of a nightingale to $\mathcal{N}$, $\mathcal{N}$ responds by naming a bird that sings, but if you call to $\mathcal{N}$ any bird that is not a nightingale, then $\mathcal{N}$ responds by naming a bird that doesn't sing. In other words, for any bird x, the bird $\mathcal{N}$x sings if and only if x is a nightingale."

"Very interesting," said Craig, who then took out his notebook and wrote down the following four conditions so he would not forget them.

Condition 1: All nightingales (of this forest) sing.

Condition 2: x′y sings if and only if xy doesn't sing.

Condition 3: x*y sings if and only if x(yy) sings.

Condition 4: $\mathcal{N}$x sings if and only if x is a nightingale.

Inspector Craig thanked Professor Baritoni warmly, took his leave, and spent the day ambling through this lovely forest. He retired early that evening and, curiously enough, solved the problem in his sleep close to morning. "Eureka!" he exclaimed, jumping out of bed. "I must see Baritoni immediately!" And so he dressed hurriedly, snatched a quick breakfast, and walked briskly in the direction of Baritoni's ornithological laboratory—an unusual thing for a well-bred

British gentleman to do without an invitation, but Craig can surely be excused, considering his state of euphoria. He turned a sharp bend and almost walked headlong into Baritoni, who was out for his morning constitutional, humming a tune from *Aïda*.

"I have solved your problem!" exclaimed Craig exuberantly. "There *is* a bird in this forest that sings but is not a nightingale."

"Wonderful!" cried Baritoni, clapping his hands in joy. "But tell me, is there any way we can actually *find* such a bird?"

"That depends," said Craig. "To begin with, if you know how to find a bird x and how to find a bird y, do you know how to find the bird xy?"

"Not necessarily," replied Baritoni. "However, if I know how to locate x and I know the *name* of y, then I can find the bird xy: I simply go over to x and call out the name of y. Then x *names* the bird xy. Once I know the name of xy, I can find it, because I can find any bird whose name I know. It might take several hours, but it can be done."

"Good enough!" said Craig. "Next, if you know the name of a bird x, can you find out the name of its spouse x′?"

"Oh, yes; I have a complete list of all the birds I know, telling me which is mated to which."

"Also," asked Craig, "if you know the name of a bird x, are you able to find the name of its associate x*?"

"Oh, yes; I have another such list."

"Finally," asked Craig, "do you know the name of this special bird $\mathcal{N}$?"

"Of course; its name is simply the letter $\mathcal{N}$."

"Good!" said Craig. "Then I believe I can lead you to a singing bird that is not a nightingale, but from what you've said, it may take several hours."

"In that case," said Baritoni eagerly, "let's start right now. We'll stop at the lab and I'll pack us a picnic lunch."

The two spent a good part of the day on their hunt, but they were amply rewarded. Toward twilight, they found themselves in a remote, lonely, and almost unknown region of the forest, and sure enough, perched on a low branch was a bird $\mathscr{G}$ singing away ever so beautifully, and $\mathscr{G}$ was definitely *not* a nightingale. In fact, the bird belonged to a species that neither Craig nor Baritoni had ever seen or heard before.

• 1 •

How did Craig know there was such a bird, and how did the two go about finding it?

Note: The bird $\mathscr{G}$ has subsequently come to be known as a *Gödelian* bird because Craig's method of finding it paralleled Gödel's method of finding a true sentence not provable in a certain axiom system. The reader interested in seeing this parallel should compare the problems of this chapter with those of chapters 14 and 15 of *The Lady or the Tiger?* The clue to the parallel is that singing birds correspond to true sentences and nightingales correspond to *provable* sentences. Thus a singing bird that is not a nightingale corresponds to a true sentence that is not provable in the axiom system under consideration.

2 • A Follow-up

The next morning Craig and Baritoni met again.

"You know," said Craig, "last night I thought of another way of finding a bird that sings but is not a nightingale. If you care to find it, we can do so, although I cannot guarantee that when we do, it might not turn out to be the same bird we found yesterday. But it may be worth a try."

Baritoni was delighted with the idea. So they spent the day in the forest and succeeded in finding a bird $\mathscr{G}_1$ that sang but was not a nightingale. As luck would have it, $\mathscr{G}_1$ turned out to be a different bird than $\mathscr{G}$, though this could not have been predicted. Can the reader explain this?

3 • The Bird Societies

Craig was enchanted with this forest and stayed for quite a while. He found out that the birds had organized several societies. A bird A is said to *represent* a set $\mathcal{S}$ of birds if for every bird x in the set $\mathcal{S}$, the bird Ax is a singing bird and for every bird x outside the set $\mathcal{S}$, the bird Ax is a nonsinging bird—in other words, for every bird x, the bird Ax sings if and only if x is a member of $\mathcal{S}$. A set of birds is called a *society* if it is represented by some bird. For example, the set of nightingales constitutes a society, because this set is represented by the bird $\mathcal{N}$.

Craig was interested in the following problem: Does the set of singing birds constitute a society? This can be answered on the basis of just Condition 2 and Condition 3 stated by Baritoni. What is the answer? Also, from just Condition 3, it can be proved that every society must either contain at least one bird that sings or lack at least one bird that doesn't sing. How is this proved, and what bearing does it have on the problem of whether the singing birds constitute a society?

SOLUTIONS

1 • They found the bird $\mathcal{G}$ in the following manner:

Baritoni already knew the name of the bird $\mathcal{N}$, hence by consulting his first list, he knew the name of $\mathcal{N}'$—the mate of $\mathcal{N}$. Then, by consulting his second list, Baritoni found the name of the bird $\mathcal{N}'^*$. To reduce clutter, let us refer to the bird $\mathcal{N}'^*$ as A. The two men next found the bird A, went up to it, and called out its own name. A responded by naming the bird AA. The two were then able to find AA. Now we prove that AA must be a bird that sings but is not a nightingale.

We let $\mathcal{G}$ be the bird AA—in other words, $\mathcal{G}$ is the bird

$\mathcal{N}'^*\mathcal{N}'^*$—and we will show that $\mathcal{G}$ sings but is not a nightingale.

The bird A has the property that for any bird x, the bird Ax sings if and only if xx is not a nightingale. The reason is: $\mathcal{N}'^*$x sings if and only if $\mathcal{N}'$(xx) sings, by Condition 3, and $\mathcal{N}'$(xx) sings if and only if $\mathcal{N}$(xx) doesn't sing, which is true if and only if xx is *not* a nightingale, because $\mathcal{N}$xx does sing if and only if xx *is* a nightingale, by Condition 4. Putting these three facts together, we see that $\mathcal{N}'^*$x sings if and only if xx is not a nightingale, and since $\mathcal{N}'^*$ is the bird A, Ax sings if and only if xx is not a nightingale.

Since it is true that for *every* bird x, the bird Ax sings if and only if xx is not a nightingale, then this is true if x is the bird A, and so AA sings if and only if AA is not a nightingale. This means that either AA sings and is not a nightingale, or AA doesn't sing and is a nightingale. However, all nightingales sing, as given in Condition 1, and so the second alternative is ruled out. Therefore AA does sing, but is not a nightingale.

The credit for this clever argument is ultimately due to Kurt Gödel.

2 • Let A_1 be the bird $\mathcal{N}^{*\prime}$, rather than $\mathcal{N}'^*$. Then A_1 is not necessarily the bird A, but it also has the property that for any bird x, the bird A_1x sings if and only if A_1x is not a nightingale. We leave the verification of this to the reader.

Then it follows by the same argument that the bird A_1A_1—call this bird $\mathcal{G}_1$—sings but is not a nightingale.

In summary, the bird $\mathcal{N}'^*\mathcal{N}'^*$ and the bird $\mathcal{N}^{*\prime}\mathcal{N}^{*\prime}$ are both birds that sing and neither is a nightingale.

3 • We will first prove on the basis of just Condition 3 that any society must either contain a singer or lack some non-singer.

Take any society $\mathcal{S}$. Then $\mathcal{S}$ is represented by some bird A. Now consider the bird A*. For any bird x, the bird A*x sings if and only if A(xx) sings, according to Condition 3. Also, A(xx) sings if and only if xx is a member of $\mathcal{S}$, because A represents $\mathcal{S}$. Therefore, A*x sings if and only if xx is a member of $\mathcal{S}$. Since this is true for every bird x, then in particular, A*A* sings if and only if A*A* is a member of $\mathcal{S}$. And so if A*A* does sing, then it is a member of $\mathcal{S}$, and hence $\mathcal{S}$ contains the singing bird A*A*. On the other hand, if A*A* doesn't sing, then A*A* is not in $\mathcal{S}$, hence $\mathcal{S}$ lacks at least one nonsinging bird—namely A*A*. This proves that every society $\mathcal{S}$ must either contain at least one singing bird or fail to contain at least one nonsinging bird.

Now, suppose the set of all singing birds formed a society; we would get the following contradiction: The set of all singing birds would be represented by some bird A. Then by Condition 2, the bird A′, the mate of A, would represent the set of all birds that *don't* sing—can you see why? This means that the set of nonsinging birds forms a society, but this is impossible, since this set neither contains a singing bird nor lacks a nonsinging bird. Therefore the set of singing birds is not represented by any bird—it is not a society.

Incidentally, the solution of this problem, together with Condition 1 and Condition 4, yields an alternative proof that there is a singing bird that is not a nightingale. Since the set of singing birds doesn't constitute a society but the set of nightingales does, by Condition 4, then the two sets are not the same. But all nightingales sing, by Condition 1, hence some singing bird is not a nightingale.