No, This is not a Circle!

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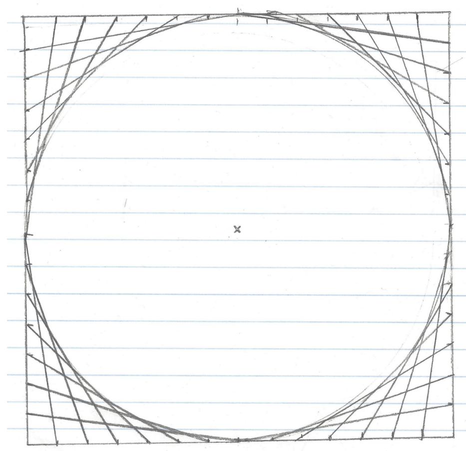

Abstract. A popular curve shown in introductory maths textbooks, seems like a circle. But it is actually a different curve. This paper discusses some elementary approaches to identify the result, including novel technological means by using GeoGebra. We demonstrate 2 ways to refute the false conjecture, 2 ways to find a correct conjecture, and 4 ways to confirm the result by proving.

All of the discussed approaches can be introduced in classrooms at various levels from middle school to high school.

Keywords: string art, envelope, GeoGebra, computer algebra, computer aided mathematics education, automated theorem proving

1 But it looks like a circle

One possible anti boredom activity is to simulate string art in a chequered notebook. This kind of activity is easy enough to do it very early, even as a child during the early school years. The resulting curve, the contour of the "strings", or more precisely, a curve whose tangents are the strings, is called an *envelope*.

According to Wikipedia [1], *an envelope of a family of curves in the plane* is a curve that is tangent to each member of the family at some point.[1] Let us assume that the investigated envelope, which is defined similarly as the learner activity in Fig. 1, is a circle. In the investigated envelope it will be assumed that a combination of 4 simple constructions is used, the axes are perpendicular, and the sums of the joined numbers are 8. To be more general, these sums may be changed to different (but fixed) numbers. These sums will be denoted by d to recall the *distance* of the origin and the furthermost point for the exterior strings.

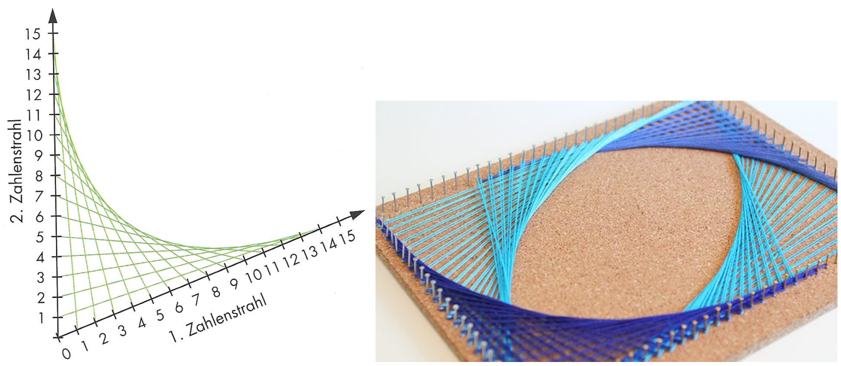

Fig. 1. An activity for young learners on the left: Join the numbers on each ray by a segment to make sums 16 [3]. Such activities are also called 'string art' when they are performed by sewing a thread on some fabric or other material. On the right there is a combinination of 4 simple constructions, but produced in a different layout than the one shown at the beginning of this paper [4].

By using the *assumption* of the circle property, in our case the family of the strings must be equally far from the center of the circle. Due to symmetry of the 4 parts of the figure, the only possible center for the circle is the midpoint of the figure. Let us consider the top-left part of the investigated figure (Fig. 2). On the left and the top the strings AB and BC have the distance $d = OA = OC$ from center O. On the other hand, the diagonal string DE has distance $OF = \frac{3}{4} \cdot d \cdot \sqrt{2}$ from the assumed center, according to the Pythagorean theorem. This latter distance is approximately $1.06 \cdot d$, that is, more than d. Consequently, the curve cannot be an exact circle. That is, it is indeed not a circle.

In schools the Pythagorean theorem is usually introduced much later than the students are ready to simply measure the length of OA and OF by using a ruler. The students need to draw, however, a large enough figure because

[1] This definition is however ambiguous: the Wikipedia page lists other non-equivalent ways to introduce the notion of envelopes. See [2] for a more detailed analysis on the various definitions.

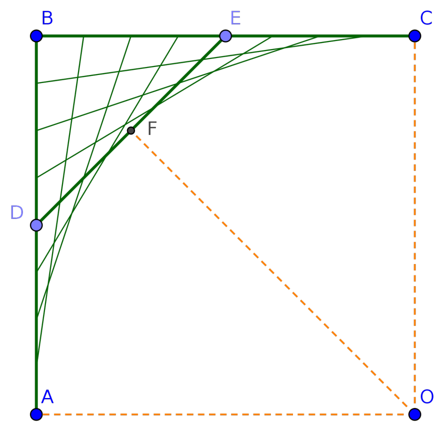

Fig. 2. Considering three strings from the family and their distance from the assumed center

the difference between OA and OF is just about 6%. Actually, both methods obviously prove that the curve is different from a circle, and the latter one can already be discussed at the beginning of the middle school.

2 OK, it is not a circle—but what is it then?

Let us continue with a possible classroom solution of the problem. Since the strings are easier to observe than the envelope, it seems logical to collect more information about the *strings*. Extending the definition of the investigated envelope by continuing the strings to both directions, we learn how the slope of the strings changes while continuing the extension more and more (Fig. 3).

The strings in the extension support the idea that the tangents of the curve, when $|n|$ is large enough, are almost parallel to the line $y = -x$. This observation may refute the opinion that the curve is eventually a hyperbola (which has two asymptotes, but they are never parallel).

On the other hand, by changing the segments in Fig. 3 to lines an obvious conjecture can be claimed, that is, the curve is a parabola (Fig. 4). Thus the observed curve must be a union of 4 parabolic arcs.

3 We have a conjecture—can we verify that?

A GeoGebra applet in Fig. 5 can explicitly compute the equation of the envelope and plot it accurately. (See [2] for a detailed survey on the currently available

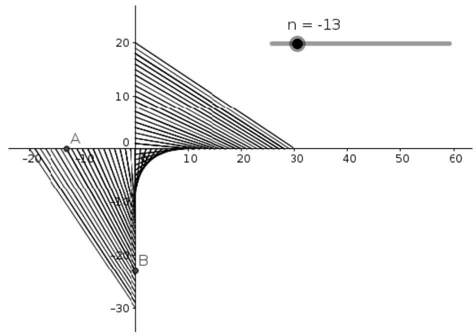

Fig. 3. Let us assume that $d = 10$ and create a GeoGebra applet as seen in the figure. (Actually, an arbitrary d can be chosen without loss of generality.) Now slider n in range $[-20, 30]$ with integer values creates points $A = (n, 0)$ and $B = (0, n - 10)$. The family of segments AB may enlighten which curve is the investigated one.

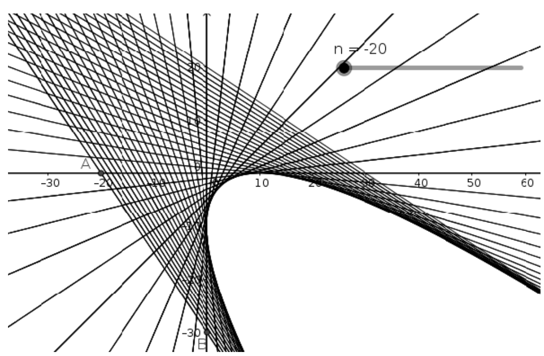

Fig. 4. Instead of segments as in Fig. 3 we use lines

software tools to visualize envelopes dynamically.) For technical reasons a slider cannot be used in this case—instead a purely Euclidean construction is required as shown in the figure. Free points A and B are defined to set the initial parameters of the applet, and finally segment $g = CC''$ describes the family of strings. The command `Envelope[`g`,`C`]` will then produce an implicit curve, which is in this concrete case $x^2 + 2xy - 20x + y^2 + 20y = -100$.

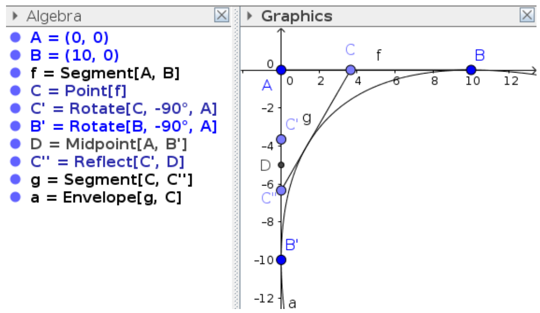

Fig. 5. A GeoGebra applet to compute and plot the parabola. Thanks to Michel Iroir and Noël Lambert for suggesting this construction method. A similar approach can be found at `http://dev.geogebra.org/trac/browser/trunk/geogebra/test/scripts/benchmark/art-plotter/tests/string-art-simple.ggb`.

GeoGebra uses heavy symbolic computations in the background to find this curve [5]. Since they are effectively done, the user may even drag points A and B to different positions and investigate the equation of the implicit curve. They are recomputed quickly enough to have an overview on the resulted curve in general—they are clearly quadratic algebraic curves in variables x and y.

Without any deeper knowledge of the classification of algebraic curves, of course, young learners cannot really decide whether the resulted curve is indeed a parabola. Advanced learners and maths teachers could however know that all real quadratic curves are either circles, ellipses, hyperbolas, parabolas, a union of two lines or a point in the plane. As in the above, we can argue that the position of the strings as tangents support only the case of parabolas here.

On the other hand, for young learners we can still find better positions for A and B. It seems quite obvious that the curve remains definitely the same (up to similarity), so it is a free choice to define the positions of A and B. By keeping A in the origin and putting B on the line $y = -x$ we can observe that the parabola is in the form $y = ax^2 + bx + c$ which is the usual way how a parabola is introduced in the classroom. (In our case actually $b = 0$.) For example, when $B = (10, -10)$, the implicit curve is $x^2 + 20y = -100$, and this can be easily converted to $y = -\frac{1}{20}x^2 - 5$ (Fig. 6).

This result is computed by using precise algebraic steps in GeoGebra. One can check these steps by examining the internal log—in this case 16 variables

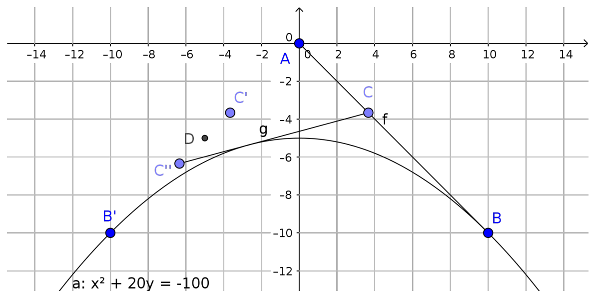

Fig. 6. By choosing $B = (10, -10)$ we obtain a simpler equation for the implicit curve

and 11 equations will be used including computing a Jacobi determinant and a Gröbner basis when eliminating all but two variables from the equation system. That is, GeoGebra actually provides a proof, albeit its steps remain hidden for the user. (Showing the detailed steps when manipulating on an equation system with so many variables makes no real sense from the educational point of view: the steps are rather mechanical and may fill hundreds of pages.)

As a conclusion, it is actually proven that the curve is a parabola. Of course, learners may want to *understand why* it that curve is.

4 Proof in the classroom

Here we provide two simple proofs on the fact that the envelope is a parabola in Fig. 6. The first method follows [6].

We need to prove that segment CC'' is always a tangent of the function $y = -\frac{1}{20}x^2 - 5$. First we compute the equation of line CC'' to find the intersection point T of CC'' and the parabola.

We recognize that if point $C = (e, -e)$, then point $C'' = (e - 10, e - 10)$.

Now we have two possible approaches to continue.

1. Since line CC'' has an equation in form $y = ax + b$, we can set up equations for points C and C'' as follows:
$$-e = a \cdot e + b \tag{1}$$
and
$$e - 10 = a \cdot (e - 10) + b. \tag{2}$$
Now (1)−(2) results in $a = 1 - \frac{1}{5}e$ and thus, by using (1) again we get $b = -2e + \frac{1}{5}e^2$.
Second, to obtain intersection point T we consider equation $ax+b = -\frac{1}{20}x^2 - 5$ which can be reformulated to search the roots of quadratic function $\frac{1}{20}x^2 + ax + b + 5$. If and only if the discriminant of this quadratic expression is zero, then CC'' is a tangent. Indeed, the determinant is $a^2 - 4 \cdot \frac{1}{20} \cdot (b+5) = a^2 - \frac{b}{5} - 1$ which is, after inserting a and b, obviously zero.

2. Another method to show that CC'' is a tangent of the parabola is to use elementary calculus. School curricula usually includes computing tangents of polynomials of the second degree.
   Let $T = (t, -\frac{1}{20}t^2 - 5)$. Now the steepness of the tangent of the parabola in T is $(-\frac{1}{20}t^2 - 5)' = -\frac{1}{10}t$. It means that the equation of the tangent is $y = -\frac{1}{10}tx + b$, here b can be computed by using $x = t$ and $y = -\frac{1}{20}t^2 - 5$, that is $b = \frac{1}{20}t^2 - 5$. The equation of the tangent is consequently

   $$y = -\frac{1}{10}tx + \frac{1}{20}t^2 - 5. \tag{3}$$

   Let us assume now that C and C'' are the intersections of the tangent and the lines $y = -x$ and $y = x$, respectively. The x-coordinate of C can be found by putting $y = -x$ in (3), it is

   $$x_C = \frac{\frac{1}{20}t^2 - 5}{\frac{1}{10}t - 1}.$$

   On the other hand, the x-coordinate of C'' can be found by putting $y = x$ in (3), it is

   $$x_{C''} = \frac{\frac{1}{20}t^2 - 5}{\frac{1}{10}t + 1}.$$

   By using some basic algebra it can be confirmed that $x_C - 10 = x_{C''}$, that is CC'' is indeed a string.

The second proof is technically longer than the first one but still achievable in many classrooms.

Both approaches are purely analytical proofs without any knowledge of the synthetic definition of a parabola. The fact is that in many classrooms, unfortunately, the synthetic definition is not introduced or even mentioned.

5 A synthetic approach

In the schools where the synthetic definition of a parabola is also introduced, the most common definition is that it is *the locus of points in the plane that are equidistant from both the directrix line ℓ and the focus point F.*

5.1 An automated answer

Without any further considerations it is possible to check (actually, *prove*) that the investigated curve is a parabola also in the symbolic sense. To achieve this result one can invoke GeoGebra's Relation Tool [7] after constructing the parabola synthetically as seen in Fig. 7.

Here the focus point F is the midpoint of BB', and the directrix line ℓ is a parallel line to BB' through A. This piece of information should be probably

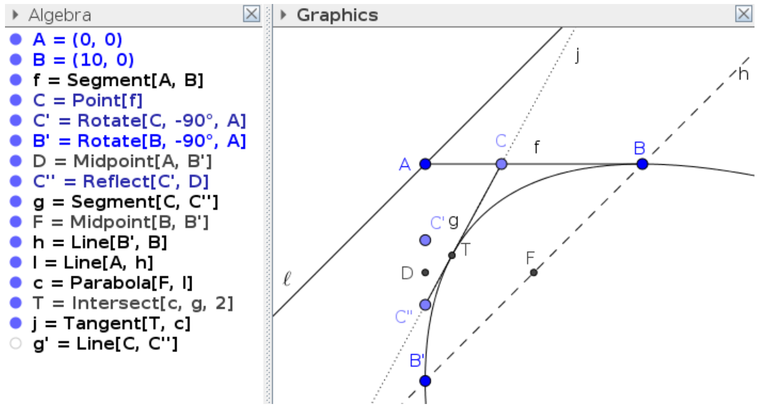

Fig. 7. A synthetic way to construct the investigated parabola in GeoGebra

kept secret by the teacher—the learners could find them on their own. Now to check if the string g is indeed a tangent of the parabola the tangent point T has been created as an intersection of the string and the parabola. Also a tangent line j has been drawn, and finally line g' which is the extension of segment g to a full line. At this point it is possible to compare g' and j by using the Relation Tool. (Note that g' is disabled for the *Graphics View* in Fig. 7 to avoid mistaking it for j. That is, by the comparison g' should be selected in the *Algebra View* on the left.)

The Relation Tool first compares the two objects numerically and reports that they are equal. By clicking the "More..." button the user obtains the symbolic result of the synthetic statement (Fig. 8).

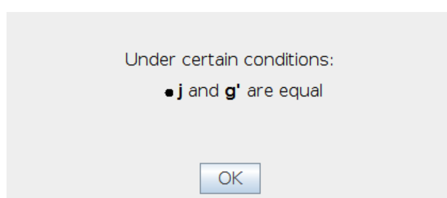

Fig. 8. Symbolic check of the synthetic statement in GeoGebra 5.0.354.0. Here "under certain conditions" mean that the statement holds in general, but there may be some extra prerequisites like avoiding degenerate positions of the free points, which cannot be further described by GeoGebra [8].

We recall that despite the construction was performed synthetically, the symbolic computations were done after translating the construction to an algebraic setup. Thus GeoGebra's internal proof is again based on algebraic equations and still hidden for the user. But in this case we indeed have a general proof for each possible construction setup, not for only one particular case as for the `Envelope` command.

5.2 A classical proof

Finally we give a classical proof to answer the original question. Here every detail uses only synthetic considerations.

The first part of the proof is a well known remark on the bisection property of the tangent. That is, by reflecting the focus point about any tangent of the parabola the mirror image is a point of the directrix line. (See e.g. [6], Sect. 3.1 for a short proof.) Clearly, it is sufficient to show that the strings have this kind of bisection property: this will result in confirming the statement.

In Fig. 9 the tangent to the parabola is denoted by j. Let F' be the mirror image of F about j. We will prove that $F' \in \ell$. Let G denote the intersection of j and FF'. Clearly $\angle CGF$ and $\angle C''GF$ are right because of the reflection.

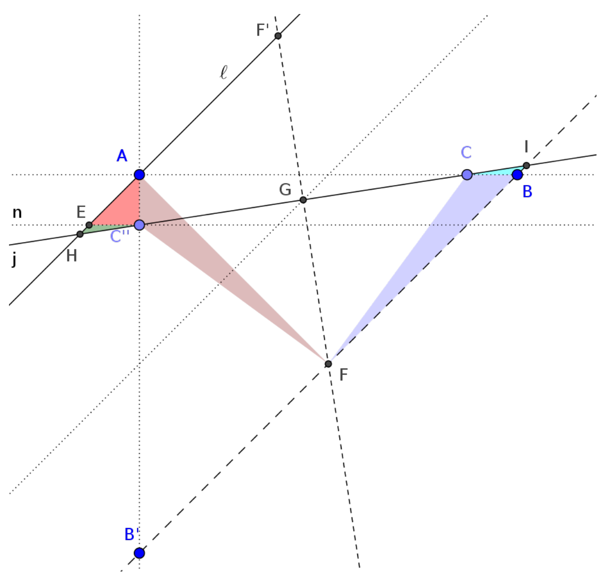

Fig. 9. A classical proof

By construction $\triangle FBC$ and $\triangle FAC''$ are congruent. Thus $FC = FC''$. Moreover, $\triangle CFC''$ is isosceles and FG is its bisector at F, in addition, $CG = C''G$.

Let n denote a parallel line to AB through C''. Let E be the intersection of ℓ and n. Also let H be the intersection of j and ℓ, and let I be the intersection of j and the line BB'. Since ℓ and BB' are parallel, moreover AB and n are also parallel, and $EC'' = CB$ (because E is actually the rotation of A around C'' by 90 degrees), we conclude that $\triangle C''EH$ and $\triangle CBI$ are congruent. This means that $IC = C''H$.

That is, using also $CG = C''G$, G must be the midpoint of HI, thus G lies on the mid-parallel of ℓ and BB'. As a consequence, reflecting F about G the resulted point F' is surely a point of line ℓ.

6 Notes

The above properties of the string art envelope are well known in the literature on Bezier curves, but usually not discussed in maths teacher trainings. The *de Casteljeau algorithm* for a Bezier curve of degree 2 is itself a proof that the curve is a parabola. (See [9, 10] for more details.)

Also among maths professionals this property seems rarely known. A recent example of a tweet of excitement is from February 2017 (Fig. 10).

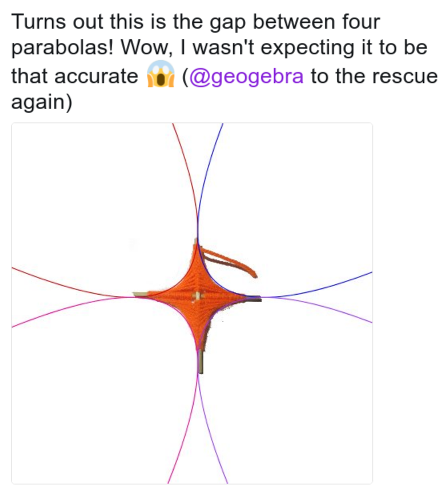

Fig. 10. Another way to experiment with the string art parabola [11]

On the other hand, our approach highlighted the classroom introduction of the string art parabola, and suggested some very recent methods by utilizing computers in the middle and high school to improve the teacher's work and the learners' skills.

Lastly, we remark that the definition of the string art envelope looks similar to the envelope of other family of lines. For example, the envelope of the sliding ladder results in a different curve, the *astroid* [1, 12, 13], a real algebraic curve of degree 6. While "physically" that is easier to construct (one just needs a ladder-like object, e. g. a pen), the geometric analysis of that is more complicated and usually involves partial derivatives. (See also [1] on a proof for identifying the string art parabola by using partial derivatives.)

7 Conclusion

An analysis of the string art envelope was presented at different levels of mathematical knowledge, by refuting a false conjecture, finding a true statement and then proving it with various means.

Discussion of a non-trivial question by using different means can give a better understanding of the problem. What is more, reasoning by visual "evidence" can be misleading, and only rigorous (or rigorous *but* computer based) proofs can be satisfactory.

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